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ADMISSIBLE PERTURBATIONS OF PROCESSES WITH INDEPENDENT INCREMENTS*

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Abstract. We investigate conditions on the law equivalence of $\boldsymbol{R}^d$-valued stochastically continuous processes with independent increments and with no Gaussian component. This problem is studied from the standpoint of perturbations. Given two processes $X = \{X(t)\}$ and $\hat{X} = \{\hat{X}(t)\}$ independent mutually, we put $X' = \{X'(t) = X(t) + \hat{X}(t)\}$. Then X' is called an *admissible perturbation* of X if X and X' induce the equivalent probability measures on the space of sample functions. The class of admissible perturbations of X is described in terms of the time-jump measures M and $\hat{M}$ associated with X and $\hat{X}$, respectively. The fine structure of this class is obtained for processes related to special infinitely divisible distributions such as stable distributions, distributions of class L and their mixtures. A simplified proof is given to the theorem of Skorokhod on the law equivalence of $\boldsymbol{R}^d$-valued processes with independent increments.

1. Introduction. The purpose of this paper is to investigate conditions on the law equivalence of multidimensional processes with nonhomogeneous independent increments. A typical example is provided by Lévy processes and processes of class L in the sense of Sato [12]. We note that probability distributions of such processes at each time are multivariate infinitely divisible distributions. We shall also focus on processes related to special infinitely divisible distributions such as stable distributions, distributions of class L and their mixtures.

Let us give some definitions and notations. Let $X = \{X(t);\ t \geqslant 0\}$ be an $\boldsymbol{R}^d$-valued stochastically continuous process with independent increments, which is defined on a basic probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. We assume that X has no Gaussian component and $X(0) = 0$ a.s. The characteristic function of $X(t)$

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can be written in the Lévy's canonical form

$$\text{(1.1)}\quad \mathbf{E}\left[\exp\left(i(z, X(t))\right)\right] = \exp\left[i(a(t), z) + \iint_{[0,t]\times \mathbf{R}_0^d} g(z, x) M(ds\,dx)\right] \quad (z \in \mathbf{R}^d,\ t \in \mathbf{T}),$$

where

$$g(z, x) = \exp\left(i(z, x)\right) - 1 - i(z, x) \mathbf{1}_D(x),$$

$$\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}, \qquad \mathbf{T} = [0, \infty), \qquad D = \{x \in \mathbf{R}_0^d;\ |x| < 1\}.$$

Here $a(t)$ is an $\mathbf{R}^d$-valued continuous function on $\mathbf{T}$ with $a(0) = 0$ and M is a Borel measure on $\mathbf{S} = \mathbf{T} \times \mathbf{R}_0^d$ such that

$$\text{(1.2)}\quad M(\{t\} \times \mathbf{R}_0^d) = 0 \ (t \in \mathbf{T}) \quad \text{and} \quad \iint_{[0,t]\times \mathbf{R}_0^d} (1 \wedge |x|^2) M(ds\,dx) < \infty \ (t \in \mathbf{T}).$$

The probability law of X is determined by the parameters a and M. We shall express this fact by the notation $X =^d [a, M]$. It should be noted that, for given a and M, we can provide a method to construct a version of X defined on some infinite product probability space (see Section 5). Without loss of generality we may assume that X induces the probability measure $\mathbf{P}_X$ on the space $\mathbf{D}(\mathbf{T})$ of $\mathbf{R}^d$-valued right-continuous functions on $\mathbf{T}$ with left-hand limits everywhere. The Borel σ-algebra on $\mathbf{S}$ is denoted by $\mathscr{B}(\mathbf{S})$. For each $U \in \mathscr{B}(\mathbf{S})$, we denote by $J(U, X)$ the random variable defined as the number of points $t \in \mathbf{T}$ satisfying the condition $\left(t, X(t) - X(t-)\right) \in U$. Then the process $\{J(U, X);\ U \in \mathscr{B}(\mathbf{S})\}$ turns out to be a Poisson random measure on $\mathbf{S}$ with intensity M. We call M the *time-jump measure* of X. The measure M is alternatively determined by the *spectral Lévy measures* ν_t $(t \in \mathbf{T})$ on $\mathbf{R}_0^d$, which are given by $\nu_t(B) = M([0, t] \times B)$ for each $B \in \mathscr{B}(\mathbf{R}_0^d)$.

Given two σ-finite measures μ and ν on a measurable space $(\mathbf{E}, \mathscr{E})$, the notation $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν. The notations $\mu \sim \nu$ and $\mu \perp \nu$ stand for equivalence (i.e., mutual absolute continuity) and singularity, respectively. The Hellinger–Kakutani distance is given by

$$\operatorname{dist}(\mu, \nu) = \left[\int_{\mathbf{E}} \{\sqrt{d\mu} - \sqrt{d\nu}\}^2\right]^{1/2}.$$

Now let $Y = \{Y(t);\ t \geqslant 0\}$ be another $\mathbf{R}^d$-valued process with independent increments and characterized by $Y =^d [b, N]$. Then we have the fundamental

THEOREM 1. *$\mathbf{P}_X \sim \mathbf{P}_Y$ if and only if the following conditions* (i)–(iii) *hold simultaneously*:

(i) $M \sim N$,

(ii) $\operatorname{dist}(M, N) < \infty$,

(iii) $a(t) - b(t) = \iint_{[0,t]\times D} x\{M - N\}(ds\,dx)$ $(t \in \mathbf{T})$.

Furthermore, if $M \sim N$, then either $\mathbf{P}_X \sim \mathbf{P}_Y$ or $\mathbf{P}_X \perp \mathbf{P}_Y$.

Theorem 1 is essentially due to Skorokhod [13]. We can give a simplified proof by using certain versions of X and Y based on Poisson random measures, which are defined on some infinite product probability spaces. The proof is then reduced to Kakutani's theorem on the equivalence of infinite product probability measures. This procedure is similar to the technique employed in our previous paper [4], and so we shall only describe the outline of the proof of Theorem 1 in the final Section 5. Since we are interested in cases where Y is a perturbation of X, it is worthwhile to reformulate this theorem in a slightly different way. Let $\hat{X} = \{\hat{X}(t); t \geq 0\}$ be an R^d-valued process with independent increments and characterized by $\hat{X} =^d [\hat{a}, \hat{M}]$. Assume that X and $\hat{X}$ are independent. Let $X' = \{X'(t); t \geq 0\}$ be another process defined by $X'(t) = X(t)+\hat{X}(t)$ $(t \geq 0)$. Then X' is an R^d-valued process with independent increments and characterized by $X' =^d [a+\hat{a}, M+\hat{M}]$. Since X' is considered as a perturbation of X, it is convenient to call $\{X, \hat{X}, X'\}$ a *perturbation triplet.* In particular, we call X' an *admissible perturbation* of X if $P_X \sim P_{X'}$. The following theorem is immediately deduced from Theorem 1.

THEOREM 2. *Let* $\{X, \hat{X}, X'\}$ *be a perturbation triplet with* $X =^d [a, M]$ *and* $\hat{X} =^d [\hat{a}, \hat{M}]$. *Then* $P_X \sim P_{X'}$ *if and only if the following conditions* (i)–(iii) *hold simultaneously*:

(i) $\hat{M} \ll M$,

(ii) $\operatorname{dist}(M, M+\hat{M}) < \infty$,

(iii) $\hat{a}(t) = \iint_{[0,t]\times D} x\hat{M}(ds\,dx)$ $(t \in T)$.

Furthermore, if $\hat{M} \ll M$, *then either* $P_X \sim P_{X'}$ *or* $P_X \perp P_{X'}$.

In general, the problem of law equivalence is reduced to the study of admissible perturbations. Suppose we are given two processes X and Y with $X =^d [a, M]$ and $Y =^d [b, N]$. Let Z and W be processes with $Z =^d [c, L]$ and $W =^d [c, L]$, where L is a time-jump measure on S satisfying $L \leq M$ and $L \leq N$. Let $\{Z, \hat{Z}, Z'\}$ and $\{W, \hat{W}, W'\}$ be perturbation triplets with $\hat{Z} =^d [a-c, \hat{L}_1]$ and $\hat{W} =^d [b-c, \hat{L}_2]$, where $\hat{L}_1 = M-L$ and $\hat{L}_2 = N-L$. Then we have $P_Z = P_W$, $P_X = P_{Z'}$ and $P_Y = P_{W'}$. Therefore, if $P_Z \sim P_{Z'}$ and $P_W \sim P_{W'}$, then $P_X \sim P_Y$. In other words, the problem of law equivalence of X and Y reduces to the study of admissible perturbations of Z and W. When we choose $L = M \wedge N$, we have the following:

(i) $M \sim N$ if and only if both $\hat{L}_1 \ll L$ and $\hat{L}_2 \ll L$;

(ii) $\operatorname{dist}(M, N) < \infty$ if and only if both $\operatorname{dist}(L, M) < \infty$ and $\operatorname{dist}(L, N) < \infty$.

Therefore, putting

$$c(t) = a(t) - \iint_{[0,t]\times D} x\hat{L}_1(ds\,dx) \qquad (t \in T),$$

we see by Theorems 1 and 2 that

(iii) $P_X \sim P_Y$ if and only if both $P_Z \sim P_{Z'}$ and $P_W \sim P_{W'}$.

Thus we are naturally led to the following

DEFINITION. Let M and $\hat{M}$ be time-jump measures defined on S. We say that $\hat{M}$ is *M-admissible* if

$$\hat{M} \ll M \quad \text{and} \quad \operatorname{dist}(M, M+\hat{M}) < \infty. \tag{1.3}$$

We denote by $\mathscr{A}d(M)$ the collection of all M-admissible time-jump measures on S.

In Section 2 we investigate general properties of $\mathscr{A}d(M)$. For this purpose we shall introduce two kinds of binomial operations among σ-finite measures on a measurable space. It should be noted that the class $\mathscr{A}d(M)$ forms a convex cone. We also consider a subclass of $\mathscr{A}d(M)$, which is determined only by the restriction of M to the subset $T \times D$ of S. Section 3 is devoted to the descriptions of typical Lévy measures related to the polar decomposition of R_0^d, which are constructed by radial and spherical components. We investigate also the non-Cartesian product case. In Section 4 the relation $\hat{M} \in \mathscr{A}d(M)$ is described precisely in terms of the radial and spherical components of the associated Lévy measures. This enables us to describe the fine structure of the class of admissible perturbations of X. Finally, in Section 5 we give the outline of the proof of Theorem 1.

2. The class $\mathscr{A}d(M)$ of M-admissible time-jump measures on S. First we introduce a function space defined on a σ-finite measure space $(E, \mathscr{E}, \nu)$. Let ϕ be a measurable function defined on $(E, \mathscr{E})$ and put

$$\|\phi\| = \left[\int_E \{|\phi| \wedge |\phi|^2\} d\nu\right]^{1/2}.$$

We denote by $L^*(E, \nu)$ the set of all measurable functions ϕ on $(E, \mathscr{E})$ with $\|\phi\| < \infty$. Then $L^*(E, \nu)$ forms a vector space with metric $d(\phi, \psi) = \|\phi - \psi\|$. For each $p \in [1, \infty]$, $L^p(E, \nu)$ denotes the L^p-space with norm $\|\phi\|_p$. The first three lemmas are simple, and so their proof is omitted.

LEMMA 1. (1) *Assume that $\phi, \psi \in L^*(E, \nu)$, $g \in L^\infty(E, \nu)$ and $a \geqslant 0$. Then*

$$\|\phi+\psi\| \leqslant \|\phi\| + \|\psi\|, \quad a(1 \wedge a)\|\phi\|^2 \leqslant \|a\phi\|^2 \leqslant a(1 \vee a)\|\phi\|^2,$$

$$\|g\phi\|^2 \leqslant \|g\|_\infty (1 \vee \|g\|_\infty)\|\phi\|^2.$$

(2) *The spaces $L^1(E, \nu)$ and $L^2(E, \nu)$ are continuously imbedded in $L^*(E, \nu)$.*

Let μ, ν, μ_j and ν_j be σ-finite measures on $(E, \mathscr{E})$. Let $U \in \mathscr{E}$ be given arbitrarily. We write $\mu_1 \sim \mu_2$ on U if the restrictions to U of μ_1 and μ_2 are equivalent. The notation $\mu_1 \approx \mu_2$ on U means that $\mu_1 \sim \mu_2$ on U, $d\mu_1/d\mu_2 \in L^\infty(U, \mu_2)$ and $d\mu_2/d\mu_1 \in L^\infty(U, \mu_1)$ hold simultaneously. Then $\mu_1 \approx \mu_2$ on U determines an equivalence relation in the set of all σ-finite measures on $(E, \mathscr{E})$. We write $\mu_1 \leqslant \mu_2$ if $\mu_1(U) \leqslant \mu_2(U)$ for each $U \in \mathscr{E}$. When $\mu(E) > 0$, we write simply $\mu > 0$.

DEFINITION. When $\mu \ll \nu$, we set

$$[\mu/\nu] = \Big[\int_E \{1 \wedge (d\mu/d\nu)\}\, d\mu\Big]^{1/2} \quad \text{and} \quad I(\mu/\nu) = \int_E \{1 \vee (d\mu/d\nu)\}\, d\mu.$$

LEMMA 2. (1) $[\mu/\nu]^2 \leqslant I(\mu/\nu) \leqslant \mu(E) + \int_E (d\mu/d\nu)\, d\mu \leqslant 2I(\mu/\nu)$.

(2) $0 < I(\mu/\nu) < \infty$ *implies* $0 < \mu(E) < \infty$.

(3) $0 < \mu(E) < \infty$ *implies* $0 < [\mu/\nu] < \infty$.

Noting that $[\mu/\nu] = \big[\int_E \{(d\mu/d\nu) \wedge (d\mu/d\nu)^2\}\, d\nu\big]^{1/2}$, we see by Lemma 1 the following

LEMMA 3. (1) *If* $\mu_1 \ll \nu$ *and* $\mu_2 \ll \nu$, *then* $[\mu_1 + \mu_2/\nu] \leqslant [\mu_1/\nu] + [\mu_2/\nu]$.

(2) *If* $\mu_1 \ll \mu_2$ *and* $\mu_2 \ll \nu$ *with* $d\mu_1/d\mu_2 \in L^\infty(E, \nu)$, *then*

$$[\mu_1/\nu]^2 \leqslant \|d\mu_1/d\mu_2\|_\infty \{1 \vee \|d\mu_1/d\mu_2\|_\infty\} [\mu_2/\nu]^2.$$

In particular, if $\mu_1 \leqslant \mu_2$ *and* $\mu_2 \ll \nu$, *then* $[\mu_1/\nu] \leqslant [\mu_2/\nu]$.

(3) *If* $\mu \ll \nu_1$ *and* $\nu_1 \ll \nu_2$ *with* $d\nu_1/d\nu_2 \in L^\infty(E, \nu_2)$, *then*

$$[\mu/\nu_2]^2 \leqslant \|d\nu_1/d\nu_2\|_\infty \{1 \vee \|d\nu_1/d\nu_2\|_\infty\} [\mu/\nu_1]^2.$$

In particular, if $\mu \ll \nu_1$ *and* $\nu_1 \leqslant \nu_2$, *then* $[\mu/\nu_2] \leqslant [\mu/\nu_1]$.

In what follows, M, $\hat{M}$, M_j and $\hat{M}_j$ denote time-jump measures defined on $S = T \times R_0^d$. We note that ϕM is also a time-jump measure on S for any $\phi \in L^\infty(S, M)$ with $\phi \geqslant 0$. We put $S(\delta) = \{(t, x) \in S;\ |x| < \delta\}$ $(\delta > 0)$. Then we have

PROPOSITION 1. (1) *Assume that* $\hat{M} \ll M$. *Then* $\hat{M} \in \mathscr{A}d(M)$ *if and only if* $[\hat{M}/M] < \infty$.

(2) *Assume that* $\hat{M} \ll M$ *and* $\int_S (1 \wedge |x|^2)\, \hat{M}(ds\, dx) < \infty$. *Then* $\hat{M} \in \mathscr{A}d(M)$ *if and only if*

$$\int_{S(\delta)} \{1 \wedge (d\hat{M}/dM)\}\, d\hat{M} < \infty \quad \textit{for some } \delta > 0.$$

Proof. First we look at the expressions

$$\operatorname{dist}(M, M + \hat{M})^2 = \int_S f(d\hat{M}/dM)\, dM \quad \text{and} \quad [\hat{M}/M]^2 = \int_S g(d\hat{M}/dM)\, dM,$$

where $f(t) = (\sqrt{t+1} - 1)^2$ and $g(t) = t \wedge t^2$ $(t \geqslant 0)$. Then the assertion (1) follows from the inequalities $g(t)/C \leqslant f(t) \leqslant Cg(t)$ on $[0, \infty)$ for some constant $C > 0$. The assertion (2) follows from $\hat{M}(S \backslash S(\delta)) < \infty$ and

$$[\hat{M}/M]^2 = \int_{S(\delta)} \{1 \wedge (d\hat{M}/dM)\}\, d\hat{M} + \int_{S\backslash S(\delta)} \{1 \wedge (d\hat{M}/dM)\}\, d\hat{M} \quad (\delta > 0).$$

On account of Proposition 1 we denote by $\mathscr{A}d^*(M)$ the collection of all $\hat{M} \in \mathscr{A}d(M)$ satisfying $\int_S (1 \wedge |x|^2)\, \hat{M}(ds\, dx) < \infty$. We note that $\mathscr{A}d(M)$ and $\mathscr{A}d^*(M)$ are convex cones. Furthermore, we see immediately by Proposition 1 and Lemma 3 the following

PROPOSITION 2. (1) *If* $\hat{M} \in \mathscr{A}d(M)$ *and* $\phi \in L^\infty(S, M)$ *with* $\phi \geqslant 0$, *then* $\phi\hat{M} \in \mathscr{A}d(M)$.

(2) *If* $\hat{M}_1, \hat{M}_2 \in \mathscr{A}d(M)$, *then* $\hat{M}_1 + \hat{M}_2 \in \mathscr{A}d(M)$.

(3) *If* $\hat{M}_1 \in \mathscr{A}d(M)$ *and* $\hat{M}_2 \in \mathscr{A}d(M + \hat{M}_1)$, *then* $\hat{M}_1 + \hat{M}_2 \in \mathscr{A}d(M)$.

PROPOSITION 3. (1) *If* $M_1 \ll M_2$ *on* S *with* $dM_1/dM_2 \in L^\infty(S(\delta), M_2)$ *for some* $\delta > 0$, *then* $\mathscr{A}d^*(M_1) \subset \mathscr{A}d^*(M_2)$.

(2) *If* $M_1 \sim M_2$ *on* S *and* $M_1 \approx M_2$ *on* $S(\delta)$ *for some* $\delta > 0$, *then* $\mathscr{A}d^*(M_1) = \mathscr{A}d^*(M_2)$.

In the light of Proposition 3, it is important to investigate the class $\mathscr{A}d(M)$ for typical time-jump measures M. Now we consider the special case, where the time-jump measures are expressed as product measures on $S = T \times R_0^d$. A Borel measure ν on R_0^d is called a *Lévy measure* if it satisfies the condition

$$\int_{R_0^d} (1 \wedge |x|^2)\, \nu(dx) < \infty. \tag{2.1}$$

ASSUMPTION (A.1). Let m and $\hat{m}$ be nonatomic Radon measures on T. Let ν and $\hat{\nu}$ be Lévy measures on R_0^d. We assume that

(i) $\hat{m} \ll m$ on T and $\hat{\nu} \ll \nu$ on R_0^d,

(ii) $M = m \times \nu$ and $\hat{M} = \hat{m} \times \hat{\nu}$ on $S = T \times R_0^d$.

PROPOSITION 4. *Suppose* M *and* $\hat{M}$ *satisfy* (A.1).

(1) *Assume that* $0 < I(\hat{m}/m) < \infty$. *Then* $\hat{M} \in \mathscr{A}d(M)$ *if and only if* $[\hat{\nu}/\nu] < \infty$.

(2) *Assume that* $[\hat{m}/m] = \infty$. *Then* $\hat{M} \in \mathscr{A}d(M)$ *if and only if* $\hat{M} = 0$.

Proof. It follows from (A.1) that $\hat{M} \ll M$. We can easily show the inequalities

$$[\hat{m}/m]^2 [\hat{\nu}/\nu]^2 \leqslant [\hat{M}/M]^2 \leqslant I(\hat{m}/m) [\hat{\nu}/\nu]^2. \tag{2.2}$$

We see by Lemma 2 that $0 < I(\hat{m}/m) < \infty$ implies $0 < [\hat{m}/m] < \infty$. Therefore, the statement (1) is seen from (2.2). Since $[\hat{\nu}/\nu] = 0$ implies $\hat{\nu} = 0$, the statement (2) follows from the first inequality of (2.2).

EXAMPLE 1 ($d = 1$). Let M and $\hat{M}$ be time-jump measures on $S = T \times R_0^1$ given by

(i) $M = m \times \nu$ and $\hat{M} = \hat{m} \times \hat{\nu}$ on $S = T \times R_0^1$,

(ii) $m(dt) = dt$ and $\hat{m}(dt) = \{t^p 1_{(0,1)}(t) + t^q 1_{[1,\infty)}(t)\}\, dt$ on T with $p > -1$ and $q \in R$,

(iii) $\nu(dx) = |x|^{-\alpha-1} dx$ and $\hat{\nu}(dx) = |x|^{-\beta-1} dx$ on R_0^1 with $\alpha, \beta \in (0, 2)$.

Then we have the following:

(1) $\hat{m}(T) < \infty$ iff $q < -1$,

(2) $[\hat{m}/m] < \infty$ iff $q < -1/2$,

(3) $I(\hat{m}/m) < \infty$ iff both $q < -1$ and $-1/2 < p$,

(4) $[\hat{\nu}/\nu] < \infty$ iff $\beta/\alpha < 1/2$,

(5) $\hat{M} \in \mathscr{A}d(M)$ iff both $\beta/\alpha < 1/2$ and $q < -1 + \beta/\alpha < p$.

DEFINITION. Let ν and $\hat{\nu}$ be Lévy measures on $\boldsymbol{R}_0^d$. Then $\hat{\nu}$ is said to be *ν-admissible* if both $\hat{\nu} \ll \nu$ and $[\hat{\nu}/\nu] < \infty$ are satisfied.

3. Lévy measures obtained by the polar decomposition of $\boldsymbol{R}_0^d$. In this section we are concerned with typical Lévy measures on $\boldsymbol{R}_0^d$, which are related to the polar decomposition of $\boldsymbol{R}_0^d$. Let h: $\boldsymbol{R}_+ \times S^{d-1} \to \boldsymbol{R}_0^d$ be a bijection defined by $h(r, \xi) = r\xi$, where $\boldsymbol{R}_+ = (0, \infty)$ and $S^{d-1} = \{x \in \boldsymbol{R}^d;\ |x| = 1\}$. Let λ be a finite Borel measure on S^{d-1}. Let q be a radial Lévy measure on $\boldsymbol{R}_+$. That is, q is a Borel measure on $\boldsymbol{R}_+$ satisfying

$$\int_{\boldsymbol{R}_+} (1 \wedge r^2)\, q(dr) < \infty. \tag{3.1}$$

Then we define a Lévy measure ν on $\boldsymbol{R}_0^d$ given by $\nu = (q \times \lambda) \circ h^{-1}$. For convenience, we write $\nu = \langle q \times \lambda \rangle$. We note that

$$\nu(B) = \int_{S^{d-1}} \lambda(d\xi) \int_{\boldsymbol{R}_+} 1_B(r\xi)\, q(dr) \quad \text{for each } B \in \mathscr{B}(\boldsymbol{R}_0^d). \tag{3.2}$$

We are interested in the case that q is absolutely continuous with respect to Lebesgue measure on $\boldsymbol{R}_+$.

TYPE (I). Let ν be a Lévy measure on $\boldsymbol{R}_0^d$ expressed in the form

(3.3) $\nu = \langle q \times \lambda \rangle$ with $q(dr) = k(r) r^{-1} dr$ and a finite Borel measure λ on S^{d-1},

where $k(r)$ is a nonnegative nonincreasing and right-continuous function on $\boldsymbol{R}_+$ such that

$$0 < \int_{\boldsymbol{R}_+} (r \wedge r^{-1}) k(r)\, dr < \infty. \tag{3.4}$$

In this case we write

$$\nu = [\langle q \times \lambda \rangle \mid k(r)].$$

TYPE (I_0). Let ν be a Lévy measure on $\boldsymbol{R}_0^d$ expressed as (3.3) with $\lambda > 0$ and

$$k(r) = \int_{(0,2)} r^{-\alpha} w(d\alpha) \quad \text{with } 0 < \int_{(0,2)} \{\alpha(2-\alpha)\}^{-1} w(d\alpha) < \infty, \tag{3.5}$$

where w is a Borel measure on $(0, 2)$. In this case we write

$$\nu = [\langle q \times \lambda \rangle \mid k(r), w(d\alpha)].$$

The function $k(r)$ given by (3.5) satisfies condition (3.4). The set of *stable distributions* on $\boldsymbol{R}^d$ with index $\alpha \in (0, 2)$ coincides with the set of infinitely divisible distributions on $\boldsymbol{R}^d$ with no Gaussian component and with Lévy measures ν expressed as (3.3), where $\lambda > 0$ and $k(r) = r^{-\alpha}$ for some $\alpha \in (0, 2)$ (see Sato [11]). Therefore, the class of Lévy measures of Type (I_0) consists of mixtures of Lévy measures of stable distributions with index $\alpha \in (0, 2)$. This class plays an important role from the standpoint of perturbations (see Section 4).

A further wide class of functions $k(r)$ can be defined as follows. Let $k(r)$ be a function on $\boldsymbol{R}_+$ given by

$$(3.6)\quad k(r) = \Phi(r^{-1}) \quad \text{with } \Phi(t) = \int_{\boldsymbol{R}_+} \{1-\exp(-t^2u)\} u^{-1} \Lambda(du) \ (t \geqslant 0),$$

where Λ is a Borel measure on $\boldsymbol{R}_+$ satisfying the condition

$$(3.7)\qquad 0 < \int_{\boldsymbol{R}_+} (1 \wedge u^{-1})\{1+\log(u \vee u^{-1})\} \Lambda(du) < \infty.$$

Then $k(r)$ is a strictly decreasing positive continuous function on $\boldsymbol{R}_+$ and satisfies condition (3.4). The function $k(r)$ in (3.5) may be expressed alternatively in the form (3.6). In particular, we have the expression $h_\alpha(r) = \Phi_\alpha(r^{-1})$ for $h_\alpha(r) = r^{-\alpha}$, where

$$\Phi_\alpha(t) = \int_{\boldsymbol{R}_+} \{1-\exp(-t^2u)\} u^{-1} \Lambda_\alpha(du), \qquad \Lambda_\alpha(du) = C_\alpha u^{-\alpha/2} du,$$

$$C_\alpha = (2/\alpha)\Gamma(1-\alpha/2) > 0 \qquad (0 < \alpha < 2).$$

When $k(r)$ is given by (3.6), the relation $\Lambda \approx \Lambda_\alpha$ on $\boldsymbol{R}_+$ implies the inequalities $h_\alpha(r)/C \leqslant k(r) \leqslant Ch_\alpha(r)$ on $\boldsymbol{R}_+$ for some constant $C > 0$. For the sake of simplicity, we denote this by the notation $k(r) \asymp h_\alpha(r)$ on $\boldsymbol{R}_+$.

PROPOSITION 5. *Assume that*
(i) $\nu = \langle q \times \lambda \rangle$ *and* $\hat{\nu} = \langle \hat{q} \times \hat{\lambda} \rangle$ *on* $\boldsymbol{R}_0^d$,
(ii) $\hat{q} \ll q$ *on* $\boldsymbol{R}_+$ *and* $\hat{\lambda} \ll \lambda$ *on* S^{d-1} *with* $0 < I(\hat{\lambda}/\lambda) < \infty$.
Then $[\hat{\nu}/\nu] < \infty$ *if and only if* $\int_{(0,\delta)} \{1 \wedge (d\hat{q}/dq)\} d\hat{q} < \infty$ *for some* $\delta > 0$.

Proof. Analogously to the case of Proposition 4, we have

$$[\hat{\lambda}/\lambda]^2 [\hat{q}/q]^2 \leqslant [\hat{\nu}/\nu]^2 \leqslant I(\hat{\lambda}/\lambda)[\hat{q}/q]^2.$$

We see by Lemma 2 that $0 < I(\hat{\lambda}/\lambda) < \infty$ implies $0 < [\hat{\lambda}/\lambda] < \infty$. Therefore $[\hat{\nu}/\nu] < \infty$ if and only if $[\hat{q}/q] < \infty$. Thus we obtain the conclusion by $\hat{q}([\delta, \infty)) < \infty$ and

$$[\hat{q}/q]^2 = \int_{(0,\delta)} \{1 \wedge (d\hat{q}/dq)\} d\hat{q} + \int_{[\delta,\infty)} \{1 \wedge (d\hat{q}/dq)\} d\hat{q} \qquad (\delta > 0).$$

EXAMPLE 2 ($d = 2$). Let ν and $\hat{\nu}$ be Lévy measures on $\boldsymbol{R}_0^2$ given by
(i) $\nu = \langle q \times \lambda \rangle$ and $\hat{\nu} = \langle \hat{q} \times \hat{\lambda} \rangle$ on $\boldsymbol{R}_0^2$,
(ii) $q(dr) = r^{-\alpha-1} dr$ and $\hat{q}(dr) = r^{-\beta-1} dr$ on $\boldsymbol{R}_+$ with $\alpha, \beta \in (0, 2)$,
(iii) $\lambda(d\theta) = d\theta$ and $\hat{\lambda}(d\theta) = \theta^p d\theta$ on $[0, 2\pi)$ with $p > -1$,
where $(r, \theta) \in \boldsymbol{R}_+ \times [0, 2\pi)$ are the polar coordinates of $(x, y) \in \boldsymbol{R}_0^2$: $x = r\cos\theta$, $y = r\sin\theta$. Then we have the following:
(1) $I(\hat{\lambda}/\lambda) < \infty$ iff $p > -1/2$,
(2) $[\hat{q}/q] < \infty$ iff $\beta/\alpha < 1/2$,
(3) $[\hat{\nu}/\nu] < \infty$ iff both $\beta/\alpha < 1/2$ and $p > -1+\beta/\alpha$.

We now look at the Lévy measures on $\boldsymbol{R}_0^d$ constructed from non-Cartesian product measures on $\boldsymbol{R}_+ \times S^{d-1}$. Let λ be a finite Borel measure on S^{d-1}. Let $\{q_\xi;\ \xi \in S^{d-1}\}$ be a family of radial Lévy measures on $\boldsymbol{R}_+$ such that $q_\xi(E)$ is a Borel measurable function in ξ for fixed $E \in \mathscr{B}(\boldsymbol{R}_+)$ and

(3.8) $$\int_{S^{d-1}} \lambda(d\xi) \int_{\boldsymbol{R}_+} (1 \wedge r^2) q_\xi(dr) < \infty \quad \text{with } q_\xi > 0 \text{ for each } \xi \in S^{d-1}.$$

Then we define a Lévy measure ν on $\boldsymbol{R}_0^d$ given by

(3.9) $$\nu(B) = \int_{S^{d-1}} \lambda(d\xi) \int_{\boldsymbol{R}_+} 1_B(r\xi) q_\xi(dr) \quad \text{for each } B \in \mathscr{B}(\boldsymbol{R}_0^d).$$

For convenience, we write $\nu = \langle \{q_\xi\} \times \lambda \rangle$. Without loss of generality, condition (3.8) may be replaced by

(3.8)* $$0 < \int_{\boldsymbol{R}_+} (1 \wedge r^2) q_\xi(dr) = C < \infty \quad \text{with } C \text{ being independent of } \xi.$$

Indeed, putting

$$Q(\xi) = \int_{\boldsymbol{R}_+} (1 \wedge r^2) q_\xi(dr), \qquad \tilde{\lambda}(d\xi) = Q(\xi)\lambda(d\xi) \quad \text{and} \quad \tilde{q}_\xi(dr) = Q(\xi)^{-1} q_\xi(dr),$$

we have an alternative expression

$$\nu(B) = \int_{S^{d-1}} \tilde{\lambda}(d\xi) \int_{\boldsymbol{R}_+} 1_B(r\xi) \tilde{q}_\xi(dr) \quad \text{for each } B \in \mathscr{B}(\boldsymbol{R}_0^d).$$

Furthermore, we see that $\tilde{\lambda}$ is finite and that condition (3.8)* holds for the family $\{\tilde{q}_\xi;\ \xi \in S^{d-1}\}$.

TYPE (II). Let ν be a Lévy measure on $\boldsymbol{R}_0^d$ expressed in the form

(3.10) $\nu = \langle \{q_\xi\} \times \lambda \rangle$ with $q_\xi(dr) = k_\xi(r) r^{-1} dr$ and a finite Borel measure λ on S^{d-1}.

Here $k_\xi(r)$ is measurable in $\xi \in S^{d-1}$, and nonnegative nonincreasing and right-continuous in $r \in \boldsymbol{R}_+$, and also satisfies the condition

(3.11) $$0 < \int_{\boldsymbol{R}_+} (r \wedge r^{-1}) k_\xi(r)\, dr = C < \infty \quad \text{with } C \text{ being independent of } \xi.$$

In this case we write

$$\nu = [\langle \{q_\xi\} \times \lambda \rangle \mid k_\xi(r)].$$

TYPE (II_0). Let ν be a Lévy measure on $\boldsymbol{R}_0^d$ expressed as (3.10) with $\lambda > 0$ and

(3.12) $$k_\xi(r) = \int_{(0,2)} r^{-\alpha} w_\xi(d\alpha) \quad \text{with } 0 < \int_{(0,2)} \{\alpha(2-\alpha)\}^{-1} w_\xi(d\alpha) = C < \infty.$$

Here $\{w_\xi;\ \xi \in S^{d-1}\}$ is a family of Borel measures on $(0, 2)$ such that $w_\xi(E)$ is a Borel measurable function in ξ for fixed $E \in \mathscr{B}((0, 2))$ and C is a constant independent of ξ. In this case we write

$$\nu = [\langle \{q_\xi\} \times \lambda \rangle \mid k_\xi(r),\ w_\xi(d\alpha)].$$

EXAMPLE 3. Let w be a Borel measure on (0, 2) satisfying the condition in (3.5) and let $f(\xi, \alpha)$ be a nonnegative Borel measurable function on $S^{d-1} \times (0, 2)$ such that

$$0 < Q(\xi) = \int_{(0,2)} f(\xi, \alpha)\{\alpha(2-\alpha)\}^{-1} w(d\alpha) < \infty \qquad \text{for each } \xi \in S^{d-1}.$$

Then putting $w_\xi(E) = Q(\xi)^{-1} \int_E f(\xi, \alpha) w(d\alpha)$ for each $E \in \mathscr{B}((0, 2))$, we have a family $\{w_\xi;\ \xi \in S^{d-1}\}$ of Borel measures on (0, 2) satisfying the condition in (3.12).

We note that the set of *distributions of class L* on $\boldsymbol{R}^d$ coincides with the set of infinitely divisible distributions on $\boldsymbol{R}^d$ with Lévy measures ν of Type (II) (see Sato [11]). The class of Lévy measures of Type (II_0) plays an important role from the standpoint of perturbations (see Section 4).

PROPOSITION 6. *Assume that*

(i) $\nu = \langle \{q_\xi\} \times \lambda \rangle$ *and* $\hat{\nu} = \langle \{\hat{q}_\xi\} \times \hat{\lambda} \rangle$ *on* $\boldsymbol{R}_0^d$,

(ii) $\hat{q}_\xi \ll q_\xi$ *on* $\boldsymbol{R}_+$ *for* $\hat{\lambda}$*-a.e.* $\xi \in S^{d-1}$ *and* $\lambda \approx \hat{\lambda}$ *on* S^{d-1} *with* $\hat{\lambda} > 0$.
Then $[\hat{\nu}/\nu] < \infty$ *if and only if*

$$\int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,\delta)} \{1 \wedge (d\hat{q}_\xi/dq_\xi)\}\, d\hat{q}_\xi < \infty \qquad \textit{for some } \delta > 0.$$

Similarly to the case of Proposition 5, we have the assertion by the expression

$$[\hat{\nu}/\nu]^2 = \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{\boldsymbol{R}_+} [1 \wedge \{(d\hat{\lambda}/d\lambda)(\xi)(d\hat{q}_\xi/dq_\xi)(r)\}]\, \hat{q}_\xi(dr).$$

4. The class of admissible perturbations of $\boldsymbol{X}$**.** In this section we shall describe the fine structure of the class $\mathscr{A}d(M)$ of M-admissible time-jump measures on $\boldsymbol{S}$. This enables us to describe the class of admissible perturbations of $\boldsymbol{X}$. Let us restate Theorem 2 in terms of $\mathscr{A}d(M)$.

THEOREM 2*. *Let* $\{\boldsymbol{X}, \hat{\boldsymbol{X}}, \boldsymbol{X}'\}$ *be a perturbation triplet with* $\boldsymbol{X} =^d [a, M]$ *and* $\hat{\boldsymbol{X}} =^d [\hat{a}, \hat{M}]$. *Then* $\boldsymbol{P}_{\boldsymbol{X}} \sim \boldsymbol{P}_{\boldsymbol{X}'}$ *if and only if*

$$\hat{M} \in \mathscr{A}d(M) \quad \textit{and} \quad \hat{a}(t) = \iint_{[0,t] \times D} x\hat{M}(ds\,dx) \ \ (t \in \boldsymbol{T}). \tag{4.1}$$

Furthermore, if $\hat{M} \ll M$, *then either* $\boldsymbol{P}_{\boldsymbol{X}} \sim \boldsymbol{P}_{\boldsymbol{X}'}$ *or* $\boldsymbol{P}_{\boldsymbol{X}} \perp \boldsymbol{P}_{\boldsymbol{X}'}$.

On account of Propositions 3 and 4, we shall impose on M and $\hat{M}$ assumption (A.1) and also the following

ASSUMPTION (A.2). (i) ν is of Type (I) and $\hat{\nu}$ is of Type (I_0):

$$\nu = [\langle q \times \lambda \rangle \mid k(r)] \quad \text{and} \quad \hat{\nu} = [\langle \hat{q} \times \hat{\lambda} \rangle \mid \hat{k}(r),\ \hat{w}(d\alpha)].$$

(ii) $0 < I(\hat{m}/m) < \infty$ and $\hat{\lambda} \ll \lambda$ on S^{d-1} with $\lambda > 0$ and $0 < I(\hat{\lambda}/\lambda) < \infty$.

ASSUMPTION (K.1). $k(r)$ is a positive nonincreasing and right-continuous function on $\boldsymbol{R}_+$ satisfying both (3.4) and $k(r) \asymp h_\gamma(r)$ on $(0, \varepsilon)$ for some $\gamma \in (0, 2)$ and $\varepsilon > 0$, where we put $h_\gamma(r) = r^{-\gamma}$.

EXAMPLE 4. If $k(r)$ is expressed as (3.5) with $w(\{\gamma\}) > 0$ and $w((\gamma, 2)) = 0$ for some $\gamma \in (0, 2)$, then $k(r)$ satisfies (K.1).

THEOREM 3. *Suppose* M *and* $\hat{M}$ *satisfy* (A.1), (A.2) *and* (K.1). *Then* $\hat{M} \in \mathscr{A}d(M)$ *if and only if*

$$\hat{w}([\gamma/2, 2)) = 0 \quad \text{and} \quad \int_{(0,\gamma)} (\gamma-\alpha)^{-1} (\hat{w} * \hat{w})(d\alpha) < \infty. \tag{4.2}$$

Proof. We see by Proposition 4 that $\hat{M} \in \mathscr{A}d(M)$ if and only if $[\hat{v}/v] < \infty$. Further, we see by Proposition 5 that $[\hat{v}/v] < \infty$ if and only if

$$\int_{(0,\delta)} \{1 \wedge (\hat{k}(r)/k(r))\} \hat{k}(r) r^{-1} dr < \infty \quad \text{for some } \delta > 0. \tag{4.3}$$

It follows from (K.1) that there exists $C > 0$ satisfying $k(r)/C \leqslant h_\gamma(r) \leqslant C k(r)$ on $(0, \varepsilon)$. Then the inequalities

$$(1 \wedge C^{-1}) \{1 \wedge (\hat{k}(r)/h_\gamma(r))\} \leqslant 1 \wedge (\hat{k}(r)/k(r)) \leqslant (1 \vee C) \{1 \wedge (\hat{k}(r)/h_\gamma(r))\}$$

hold on $(0, \varepsilon)$. Therefore (4.3) is equivalent to the following

$$\int_{(0,\delta)} \{1 \wedge (\hat{k}(r)/h_\gamma(r))\} \hat{k}(r) r^{-1} dr \equiv I(\hat{k}, \delta) < \infty \quad \text{for some } \delta > 0. \tag{4.4}$$

Suppose (4.4) is true. If we assume $\hat{w}([\gamma, 2)) > 0$, we can find $\delta^* \in (0, \delta)$ and $C^* > 0$ satisfying $\hat{k}(r) \geqslant C^* h_\gamma(r)$ on $(0, \delta^*)$. Then we obtain $I(\hat{k}, \delta) = \infty$ since

$$I(\hat{k}, \delta) \geqslant (1 \wedge C^*) \int_{(0,\delta^*)} \hat{k}(r) r^{-1} dr = \infty.$$

This contradiction yields $\hat{w}([\gamma, 2)) = 0$. Then we can find $\delta^{**} \in (0, \delta)$ satisfying $\hat{k}(r) < h_\gamma(r)$ on $(0, \delta^{**})$. Therefore we obtain

$$\int_{(0,1)} \hat{k}(r)^2 r^{\gamma-1} dr \equiv J(\hat{k}) < \infty, \tag{4.5}$$

since we have

$$\int_{(0,\delta^{**})} \hat{k}(r)^2 r^{\gamma-1} dr = \int_{(0,\delta^{**})} \{\hat{k}(r)/h_\gamma(r)\} \hat{k}(r) r^{-1} dr \leqslant I(\hat{k}, \delta) < \infty.$$

By seeing that (4.5) implies (4.4), we obtain the equivalence of (4.4) and (4.5). Thus it suffices to prove the equivalence of (4.2) and (4.5). We assume that (4.5) is true. Then we see that $\hat{w}([\gamma/2, 2)) = 0$ by the inequalities

$$\infty > J(\hat{k}) \geqslant \int_{(0,1)} \{\hat{w}([\gamma/2, 2)) r^{-\gamma/2}\}^2 r^{\gamma-1} dr = \{\hat{w}([\gamma/2, 2))\}^2 \int_{(0,1)} r^{-1} dr.$$

It follows that

$$(4.6)\quad J(\hat{k}) = \int_{(0,1)} \Big[\Big\{\int_{(0,\gamma/2)} r^{-\alpha}\,\hat{w}(d\alpha)\Big\}\Big\{\int_{(0,\gamma/2)} r^{-\beta}\,\hat{w}(d\beta)\Big\}\Big] r^{\gamma-1}\,dr$$

$$= \iint_{(0,\gamma/2)^2} \Big\{\int_{(0,1)} r^{\gamma-(\alpha+\beta)-1}\,dr\Big\}\,\hat{w}(d\alpha)\,\hat{w}(d\beta)$$

$$= \iint_{(0,\gamma/2)^2} \{\gamma-(\alpha+\beta)\}^{-1}\,\hat{w}(d\alpha)\,\hat{w}(d\beta) = \int_{(0,\gamma)} (\gamma-\alpha)^{-1}\,(\hat{w}*\hat{w})(d\alpha).$$

Therefore we see that (4.2) is true. By using (4.6) again, we immediately see that (4.2) implies (4.5). The equivalence of (4.2) and (4.5) is thus proved, which completes the proof.

Corollary 1. *Suppose M and $\hat{M}$ satisfy the conditions stated in Theorem* 3. *Then $\hat{M} \in \mathscr{A}d(M)$ holds if*

$$(4.7)\qquad \hat{w}([\gamma/2,\,2)) = 0 \quad \textit{and} \quad \int_{(0,\gamma/2)} (\gamma/2-\alpha)^{-1}\,\hat{w}(d\alpha) < \infty.$$

In particular, $\hat{M} \in \mathscr{A}d(M)$ holds if $\hat{w}((\gamma/2-\delta,\,2)) = 0$ for some $\delta > 0$.

Proof. By using the inequality $(x+y)^{-1} \leqslant x^{-1}+y^{-1}$ $(x,\,y>0)$, we see by (4.6) and (4.7) that

$$\int_{(0,\gamma)} (\gamma-\alpha)^{-1}\,(\hat{w}*\hat{w})(d\alpha) \leqslant \iint_{(0,\gamma/2)^2} \{(\gamma/2-\alpha)^{-1} + (\gamma/2-\beta)^{-1}\}\,\hat{w}(d\alpha)\,\hat{w}(d\beta)$$

$$= 2\hat{w}((0,\,\gamma/2)) \int_{(0,\gamma/2)} (\gamma/2-\alpha)^{-1}\,\hat{w}(d\alpha) < \infty.$$

Therefore we obtain $\hat{M} \in \mathscr{A}d(M)$ by Theorem 3. The last part follows from the inequality

$$\int_{(0,\gamma/2)} (\gamma/2-\alpha)^{-1}\,\hat{w}(d\alpha) = \int_{(0,\gamma/2-\delta]} (\gamma/2-\alpha)^{-1}\,\hat{w}(d\alpha) \leqslant \delta^{-1}\,\hat{w}((0,\,\gamma/2-\delta]) < \infty.$$

Assumption (A.3). (i) ν is of Type (I) and $\hat{\nu}$ is of Type (II_0):

$$\nu = [\langle q\times\lambda\rangle \mid k(r)] \quad \text{and} \quad \hat{\nu} = [\langle\{\hat{q}_\xi\}\times\hat{\lambda}\rangle \mid \hat{k}_\xi(r),\,\hat{w}_\xi(d\alpha)].$$

(ii) $0 < I(\hat{m}/m) < \infty$ and $\lambda \approx \hat{\lambda}$ on S^{d-1} with $\lambda > 0$.

Assumption (K.2). There exists $\hat{\varepsilon} > 0$ satisfying $\hat{k}_\xi(r) \leqslant k(r)$ on $(0,\,\hat{\varepsilon})$ for $\hat{\lambda}$-a.e. $\xi \in S^{d-1}$.

Theorem 4. *Suppose M and $\hat{M}$ satisfy* (A.1), (A.3), (K.1) *and* (K.2). *Then $\hat{M} \in \mathscr{A}d(M)$ if and only if*

$$(4.8)\qquad \hat{w}_\xi([\gamma/2,\,2)) = 0 \ \textit{for}\ \hat{\lambda}\textit{-a.e.}\ \xi \in S^{d-1}$$

$$\textit{and} \quad \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,\gamma)} (\gamma-\alpha)^{-1}\,(\hat{w}_\xi*\hat{w}_\xi)(d\alpha) < \infty.$$

Proof. We see by Proposition 4 that $\hat{M} \in \mathscr{A}d(M)$ if and only if $[\hat{\nu}/\nu] < \infty$. We also see by Proposition 6 that $[\hat{\nu}/\nu] < \infty$ if and only if

$$(4.9)\quad \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,\delta)} \{1 \wedge (\hat{k}_\xi(r)/k(r))\} \hat{k}_\xi(r) r^{-1} dr < \infty \qquad \text{for some } \delta > 0.$$

We see by (K.2) that (4.9) is equivalent to the condition

$$(4.10)\qquad \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,\delta)} \hat{k}_\xi(r)^2 r^{\gamma-1} dr < \infty \qquad \text{for some } \delta > 0.$$

Analogously to the case of Theorem 3, we see that $\hat{w}_\xi([\gamma/2, 2)) = 0$ implies the identity

$$(4.11)\qquad \int_{(0,1)} \hat{k}_\xi(r)^2 r^{\gamma-1} dr = \int_{(0,\gamma)} (\gamma-\alpha)^{-1} (\hat{w}_\xi * \hat{w}_\xi)(d\alpha).$$

Therefore (4.8) implies (4.10). Now we assume that (4.10) is true. Then we have

$$\int_{(0,\delta)} \hat{k}_\xi(r)^2 r^{\gamma-1} dr < \infty \qquad \text{for } \hat{\lambda}\text{-a.e. } \xi \in S^{d-1}.$$

Further we see that $\hat{w}_\xi([\gamma/2, 2)) = 0$ holds for $\hat{\lambda}$-a.e. $\xi \in S^{d-1}$ by the same discussion as in the proof of Theorem 3. Thus we obtain (4.8) by using (4.11) again, which completes the proof.

Similarly to the proof of Corollary 1, we can deduce from Theorem 4 the following

Corollary 2. *Suppose M and $\hat{M}$ satisfy the conditions stated in Theorem 4. Then $\hat{M} \in \mathscr{A}d(M)$ holds if*

$$(4.12)\quad \hat{w}_\xi([\gamma/2, 2)) = 0 \text{ for } \hat{\lambda}\text{-a.e. } \xi \in S^{d-1}$$

$$\text{and} \quad \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,\gamma/2)} (\gamma/2-\alpha)^{-1} \hat{w}_\xi(d\alpha) < \infty.$$

In particular, $\hat{M} \in \mathscr{A}d(M)$ holds if $\hat{w}_\xi((\gamma/2-\delta, 2)) = 0$ for $\hat{\lambda}$-a.e. $\xi \in S^{d-1}$ with $\delta > 0$ being independent of ξ.

We now proceed to the case where Lévy measures are assumed to be of Type (II).

Assumption (A.4). (i) Both ν and $\hat{\nu}$ are of Type (II):

$$\nu = [\langle\{q_\xi\} \times \lambda\rangle \mid k_\xi(r)] \quad \text{and} \quad \hat{\nu} = [\langle\{\hat{q}_\xi\} \times \hat{\lambda}\rangle \mid \hat{k}_\xi(r)].$$

(ii) $0 < I(\hat{m}/m) < \infty$ and $\lambda \approx \hat{\lambda}$ on S^{d-1} with $\lambda > 0$.

Assumption (K.3). There exist $\varepsilon > 0$, $C > 0$ and a pair $\{\phi(\xi), \hat{\phi}(\xi)\}$ of measurable functions on S^{d-1} with $0 < \phi(\xi) < 2$ and $0 < \hat{\phi}(\xi) < 2$ such that

$$r^{-\phi(\xi)}/C \leqslant k_\xi(r) \leqslant Cr^{-\phi(\xi)} \quad \text{and} \quad r^{-\hat{\phi}(\xi)}/C \leqslant \hat{k}_\xi(r) \leqslant Cr^{-\hat{\phi}(\xi)}$$

for each $(r, \xi) \in (0, \varepsilon) \times S^{d-1}$.

EXAMPLE 5. Let $\{w_\xi;\ \xi \in S^{d-1}\}$ be a family of Borel measures on (0, 2) stated in Example 3. Assume that $f(\xi, \alpha) = 1_{(0,\phi(\xi)]}(\alpha)$ and $w(\{\phi(\xi)\}) \geqslant c$ for each $\xi \in S^{d-1}$, where $\phi(\xi)$ is a measurable function on S^{d-1} with $0 < \phi(\xi) < 2$ and c is a positive constant independent of ξ. Then $k_\xi(r) = \int_{(0,2)} r^{-\alpha} w_\xi(d\alpha)$ satisfies (K.3).

THEOREM 5. *Suppose M and $\hat{M}$ satisfy* (A.1), (A.4) *and* (K.3). *Then $\hat{M} \in \mathscr{A}d(M)$ if and only if*

$$\text{(4.13)} \qquad \hat{\phi}(\xi) < \phi(\xi)/2 \text{ for } \hat{\lambda}\text{-a.e. } \xi \in S^{d-1} \quad \text{and} \quad \int_{S^{d-1}} \{\phi(\xi)/2 - \hat{\phi}(\xi)\}^{-1} \hat{\lambda}(d\xi) < \infty.$$

Proof. By Propositions 4 and 6, we see that $\hat{M} \in \mathscr{A}d(M)$ if and only if

$$\text{(4.14)} \qquad \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,\delta)} \{1 \wedge (\hat{k}_\xi(r)/k_\xi(r))\} \hat{k}_\xi(r) r^{-1} dr < \infty \qquad \text{for some } \delta > 0.$$

By the assumption we can find $\varepsilon^* \in (0, 1)$ and $C^* > 0$ such that

$$r^{\phi(\xi)-\hat{\phi}(\xi)}/C^* \leqslant \hat{k}_\xi(r)/k_\xi(r) \leqslant C^* r^{\phi(\xi)-\hat{\phi}(\xi)} \qquad \text{for each } (r, \xi) \in (0, \varepsilon^*) \times S^{d-1}.$$

Then it is easy to see that (4.14) is equivalent to

$$\text{(4.15)} \qquad \int_{S^{d-1}} \hat{\lambda}(d\xi) \int_{(0,1)} [1 \wedge \{r^{\phi(\xi)-\hat{\phi}(\xi)}\}] r^{-\hat{\phi}(\xi)-1} dr < \infty.$$

Putting $E = \{\xi \in S^{d-1};\ \hat{\phi}(\xi) < \phi(\xi)\}$, we see that (4.15) is equivalent to the condition

$$\text{(4.16)} \qquad \hat{\lambda}(S^{d-1} \backslash E) = 0 \quad \text{and} \quad \int_E \hat{\lambda}(d\xi) \int_{(0,1)} r^{\phi(\xi)-2\hat{\phi}(\xi)-1} dr < \infty.$$

Further (4.16) is equivalent to the condition

$$\text{(4.17)} \qquad \hat{\lambda}(S^{d-1} \backslash F) = 0 \quad \text{and} \quad \int_F \{\phi(\xi) - 2\hat{\phi}(\xi)\}^{-1} \hat{\lambda}(d\xi) < \infty$$

with $F = \{\xi \in S^{d-1};\ \hat{\phi}(\xi) < \phi(\xi)/2\}$. The proof is thus completed.

In the rest of this section we are concerned with $\boldsymbol{R}^d$-valued rotationally invariant processes with independent increments. Let $X = \{X(t);\ t \geqslant 0\}$ be an $\boldsymbol{R}^d$-valued process with independent increments. Assume that the characteristic function of $X(t)$ is expressed in the form

$$\text{(4.18)} \qquad \mathbf{E}\left[\exp\left(i(z, X(t))\right)\right] = \exp\left[-m([0, t]) \int_{(0,2)} |z|^\alpha v(d\alpha)\right] \qquad (z \in \boldsymbol{R}^d,\ t \in \boldsymbol{T}),$$

where m is a nonatomic Radon measure on $\boldsymbol{T}$ and v is a finite Borel measure on (0, 2) with $v > 0$. Let us introduce a continuous function

$F(\alpha)$ on $(0, 2)$ given by

(4.19) $$F(\alpha) = g(\alpha) \int_{S^{d-1}} |(\zeta, \xi)|^{\alpha} \lambda_0(d\xi) \quad (0 < \alpha < 2),$$

$$g(\alpha) = -\Gamma(-\alpha)\cos(\pi\alpha/2) \ (\alpha \neq 1), \quad g(1) = \pi/2,$$

where λ_0 denotes the uniform probability measure on S^{d-1} and $\zeta \in S^{d-1}$. We note that $F(\alpha)$ is independent of $\zeta \in S^{d-1}$ and $F(\alpha) \asymp \{\alpha(2-\alpha)\}^{-1}$ on $(0, 2)$. Then the probability law of X is also characterized by $X =^d [0, M]$, where we set

(4.20) $$M = m \times v \text{ on } \boldsymbol{S} = \boldsymbol{T} \times \boldsymbol{R}_0^d, \quad v = [\langle q \times \lambda_0 \rangle \mid k(r), w(d\alpha)] \text{ on } \boldsymbol{R}_0^d,$$

$$w(d\alpha) = \{F(\alpha)\}^{-1} v(d\alpha) \quad \text{on } (0, 2).$$

The process X is *rotationally invariant* in the sense that the probability laws of X and $X_T = \{TX(t); t \geqslant 0\}$ coincide for each orthogonal $(d \times d)$-matrix T. Now let us consider the perturbation triplet $\{X, \hat{X}, X'\}$, where $\hat{X}$ is characterized by

(4.21) $$\mathrm{E}\left[\exp\left(i\left(z, \hat{X}(t)\right)\right)\right]$$

$$= \exp\left[-\hat{m}([0, t]) \int_{(0,2)} |z|^{\alpha} \hat{v}(d\alpha)\right] \quad (z \in \boldsymbol{R}^d, t \in \boldsymbol{T}).$$

We assume that $\hat{m}$ is a nonatomic Radon measure on $\boldsymbol{T}$ and $\hat{v}$ is a finite Borel measure on $(0, 2)$ with $\hat{v} > 0$. Then we easily see from Proposition 4 and Theorem 3 the following

THEOREM 6. *Let $\{X, \hat{X}, X'\}$ be the perturbation triplet given by* (4.18) *and* (4.21). *Assume that $v(\{\gamma\}) > 0$ with $v((\gamma, 2)) = 0$ for some $\gamma \in (0, 2)$ and $\hat{m} \ll m$ on $\boldsymbol{T}$.*

(1) *Either $P_X \sim P_{X'}$ or $P_X \perp P_{X'}$ holds.*

(2) *Assume that $0 < I(\hat{m}/m) < \infty$. Then $P_X \sim P_{X'}$ if and only if*

(4.22) $$\hat{v}([\gamma/2, 2)) = 0 \quad \textit{and} \quad \int_{(0,\gamma)} (\gamma-\alpha)^{-1} (\hat{v} * \hat{v})(d\alpha) < \infty.$$

(3) *$[\hat{m}/m] = \infty$ implies $P_X \perp P_{X'}$.*

5. The outline of the proof of Theorem 1.

5.1. *A construction of processes with independent increments.* Let X be an $\boldsymbol{R}^d$-valued process with independent increments and characterized by $X =^d [a, M]$. The aim of this subsection is to construct a version of X, which is based on a Poisson random measure and defined on some infinite product probability space associated with a certain decomposition of M. Suppose that $a(t)$ is an $\boldsymbol{R}^d$-valued continuous function on $\boldsymbol{T} = [0, \infty)$ with $a(0) = 0$ and M is a Borel measure on $\boldsymbol{S} = \boldsymbol{T} \times \boldsymbol{R}_0^d$ satisfying condition (1.2). Let us consider a decomposition $\boldsymbol{S} = \bigcup_{n=1}^{\infty} \boldsymbol{S}_n$ given by

(5.1) $$\boldsymbol{S}_n = \boldsymbol{R}_n \backslash \boldsymbol{R}_{n-1}, \quad \boldsymbol{R}_n = [0, j_n] \times \boldsymbol{B}_n$$

$$\text{and} \quad \boldsymbol{B}_n = \{x \in \boldsymbol{R}_0^d;\ |x| \geqslant \varepsilon_n\} \quad (n \geqslant 1)$$

with $R_0 = \varnothing$, where $\{\varepsilon_n\}$ is a decreasing sequence of positive numbers tending to 0 and $\{j_n\}$ is an increasing sequence of positive numbers tending to ∞. Then putting $M_n(U) = M(U \cap S_n)$ for each $U \in \mathscr{B}(S)$, we have a sequence $\{M_n;\ n \geqslant 1\}$ of finite measures on $(S, \mathscr{B}(S))$. Without loss of generality, we may assume that $M_n(S) > 0$ holds for each $n \geqslant 1$. For each $n \geqslant 1$ and $k \geqslant 1$, let $(S^k, \mathscr{B}(S^k), P_{nk})$ be the probability space defined by $P_{nk} = M_n(S)^{-k} M_n^k$, where $(S^k, \mathscr{B}(S^k), M_n^k)$ is the k-fold product of $(S, \mathscr{B}(S), M_n)$. For convenience we also define the trivial probability space $(S^0, \mathscr{B}(S^0), P_{n0})$ given by $S^0 = \{0\}$ and $\mathscr{B}(S^0) = \{\varnothing, S^0\}$. Further we consider a sequence of probability spaces $(\Omega^*;\ \mathscr{F}^*, P_n^*)$ $(n \geqslant 1)$ defined by

$$(5.2)\qquad \Omega^* = \bigcup_{k=0}^{\infty} S^k, \qquad \mathscr{F}^* = \{A^* = \bigcup_{k=0}^{\infty} A_k;\ A_k \in \mathscr{B}(S^k)\ (k \geqslant 0)\},$$

$$P_n^*(A^*) = \exp\left(-M_n(S)\right) \sum_{k=0}^{\infty} (k!)^{-1} M_n(S)^k P_{nk}(A_k)$$

for $A^* = \bigcup_{k=0}^{\infty} A_k \in \mathscr{F}^*$. Then we define the following infinite product probability space:

$$(5.3)\qquad (\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}) = \prod_{n=1}^{\infty} (\Omega^*, \mathscr{F}^*, P_n^*).$$

DEFINITION. We call $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ the *canonical probability space* associated with decomposition $M = \sum_{n=1}^{\infty} M_n$ on $(S, \mathscr{B}(S))$.

We now proceed to the construction of random measures on S, which are defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$. Let $\mathscr{N} = \mathscr{N}(S)$ be the totality of nonnegative (possibly infinite) integer-valued measures on $(S, \mathscr{B}(S))$. Let $\mathscr{F}^+(S)$ be the set of all nonnegative measurable functions on $(S, \mathscr{B}(S))$. We put

$$f^*(\mu) = \langle \mu, f \rangle = \int_S f\, d\mu \qquad \text{for } f \in \mathscr{F}^+(S) \text{ and } \mu \in \mathscr{N}.$$

Then we consider a measurable space $(\mathscr{N}, \mathscr{B}(\mathscr{N}))$, where $\mathscr{B}(\mathscr{N})$ denotes the σ-algebra on $\mathscr{N}$ generated by the family $\{f^*; f \in \mathscr{F}^+(S)\}$ of functions on $\mathscr{N}$. Let Φ: $\Omega^* \to \mathscr{N}$ be a measurable map given by $\Phi(0) = 0$ and

$$\langle \Phi(\omega^*), f \rangle = \sum_{j=1}^{k} f\left(p_j(\omega^*)\right)$$

for $f \in \mathscr{F}^+(S)$ and $\omega^* = \left(p_1(\omega^*), \ldots, p_k(\omega^*)\right) \in S^k$ $(k \geqslant 1)$. We mean by π_n: $\tilde{\Omega} \to \Omega^*$ the n-th projection map given by $\pi_n(\tilde{\omega}) = \omega_n^*$ for $\tilde{\omega} = (\omega_1^*, \omega_2^*, \ldots) \in \tilde{\Omega} = (\Omega^*)^\infty$. Then defining $(\Phi \circ \pi_n)(\tilde{\omega}) = \Phi\left(\pi_n(\tilde{\omega})\right)$ $(\tilde{\omega} \in \tilde{\Omega},\ n \geqslant 1)$, we have a sequence of independent $\mathscr{N}$-valued random elements $\Phi \circ \pi_n$ $(n \geqslant 1)$ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$. Each $\Phi \circ \pi_n$ is a Poisson random measure on S with intensity M_n and also $\Psi_n = \sum_{j=1}^{n} (\Phi \circ \pi_j)$ is a Poisson random measure on S with intensity $M_{(n)}$, where we put $M_{(n)}(U) = M(U \cap R_n)$ for $U \in \mathscr{B}(S)$. Furthermore, if we define

$$(5.4)\qquad \Psi(\tilde{\omega}) = \sum_{n=1}^{\infty} (\Phi \circ \pi_n)(\tilde{\omega}) \qquad (\tilde{\omega} \in \tilde{\Omega}),$$

then we have the following

LEMMA 4. *The $\mathcal{N}$-valued random element Ψ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}})$ is a Poisson random measure on $\boldsymbol{S}$ with intensity M. In other words, the Laplace transform of Ψ is expressed in the form*

$$(5.5)\qquad \tilde{\mathbf{E}}[\exp(-\langle \Psi, f\rangle)] = \exp[-\iint_{\boldsymbol{S}} \{1-\exp(-f(s, x))\}\, M(ds\,dx)]$$

for $f \in \mathscr{F}^+(\boldsymbol{S})$, where $\tilde{\mathbf{E}}[\cdot]$ stands for the expectation with respect to $\tilde{\boldsymbol{P}}$.

We next introduce a sequence of $\boldsymbol{R}^d$-valued processes $W_n = \{W_n(t);\ t \geqslant 0\}$ $(n \geqslant 1)$ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}})$, which are given by

$$(5.6)\qquad W_n(t, \tilde{\omega}) = a(t) + \iint_{[0,t]\times \boldsymbol{R}_0^d} x\Psi_n(ds\,dx, \tilde{\omega}) - \iint_{[0,t]\times D} xM_{(n)}(ds\,dx)$$

for $t \in \boldsymbol{T}$ and $\tilde{\omega} \in \tilde{\Omega}$. Then we have $W_n =^d [a, M_{(n)}]$. Consequently, the following lemma provides the desired version of $\boldsymbol{X}$.

LEMMA 5. *There exists an $\boldsymbol{R}^d$-valued process $W = \{W(t);\ t \geqslant 0\}$ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}})$ and characterized by $W =^d [a, M]$ such that the following conditions are satisfied:*

(i) *Almost all sample functions of W belong to the space $\mathbf{D}(\boldsymbol{T})$.*

(ii) *For almost all $\tilde{\omega} \in \tilde{\Omega}$, a sequence of sample functions $W_n(t, \tilde{\omega})$ converges to $W(t, \tilde{\omega})$ uniformly on $[0, l]$ for each $l > 0$ as n tends to ∞.*

5.2. *The "if" part of Theorem* 1. Suppose the conditions (i)–(iii) in Theorem 1 hold simultaneously. Let N_n $(n \geqslant 1)$ be a sequence of finite measures on $(\boldsymbol{S}, \mathscr{B}(\boldsymbol{S}))$ defined by $N_n(U) = N(U \cap \boldsymbol{S}_n)$ for $U \in \mathscr{B}(\boldsymbol{S})$, where $\boldsymbol{S}_n$ $(n \geqslant 1)$ are given by (5.1). By choosing $\{\varepsilon_n\}$ and $\{j_n\}$ appropriately, we may assume that both $M_n(\boldsymbol{S}) > 0$ and $N_n(\boldsymbol{S}) > 0$ hold for each $n \geqslant 1$. According to the procedure stated above, we construct the canonical probability space

$$(5.7)\qquad (\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{Q}}) = \prod_{n=1}^{\infty} (\Omega^*, \mathscr{F}^*, \boldsymbol{Q}_n^*)$$

associated with decomposition $N = \sum_{n=1}^{\infty} N_n$ on $(\boldsymbol{S}, \mathscr{B}(\boldsymbol{S}))$. We also consider a sequence of $\boldsymbol{R}^d$-valued processes $V_n = \{V_n(t);\ t \geqslant 0\}$ $(n \geqslant 1)$ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{Q}})$, which are given by

$$(5.8)\qquad V_n(t, \tilde{\omega}) = b(t) + \iint_{[0,t]\times \boldsymbol{R}_0^d} x\Psi_n(ds\,dx, \tilde{\omega}) - \iint_{[0,t]\times D} xN_{(n)}(ds\,dx)$$

for $t \in \boldsymbol{T}$ and $\tilde{\omega} \in \tilde{\Omega}$. Here we put $N_{(n)}(U) = N(U \cap \boldsymbol{R}_n)$ for $U \in \mathscr{B}(\boldsymbol{S})$. Then we have $V_n =^d [b, N_{(n)}]$. Further there exists an $\boldsymbol{R}^d$-valued process $V = \{V(t);$ $t \geqslant 0\}$ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{Q}})$ and characterized by $V =^d [b, N]$ such that, for almost all $\tilde{\omega}\in\tilde{\Omega}$, a sequence of sample functions $V_n(t,\tilde{\omega})$ converges to $V(t,\tilde{\omega})$ uniformly on $[0, l]$ for each $l>0$ as n tends to ∞. Now we see by the condition (i) that $M_n\sim N_n$ and $\boldsymbol{P}_n^*\sim \boldsymbol{Q}_n^*$ for each $n\geqslant 1$. Therefore we see by (ii) that

$$\prod_{n=1}^{\infty}\int_{\Omega^*}(d\boldsymbol{P}_n^*/d\boldsymbol{Q}_n^*)^{1/2}\,d\boldsymbol{Q}_n^*=\exp\left[-(1/2)\operatorname{dist}(M,N)^2\right]>0.$$

Thus we obtain $\tilde{\boldsymbol{P}}\sim\tilde{\boldsymbol{Q}}$ by Kakutani's theorem on the equivalence of infinite product probability measures. Therefore we can find an $\boldsymbol{R}^d$-valued function $\Theta(t,\tilde{\omega})$ on $\boldsymbol{T}\times\tilde{\Omega}$ such that the condition

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant l}|W_n(t,\tilde{\omega})-\Theta(t,\tilde{\omega})|=0\qquad\text{for each } l>0$$

holds almost surely with respect to both $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{Q}}$. Then the process $\boldsymbol{\Theta}=\{\Theta(t,\tilde{\omega});\,t\geqslant 0\}$ is characterized by

(5.9) $\boldsymbol{\Theta}=^d[a, M]$ with respect to $\tilde{\boldsymbol{P}}$, $\quad\boldsymbol{\Theta}=^d[c, N]$ with respect to $\tilde{\boldsymbol{Q}}$,

$$c(t)=a(t)-\iint_{[0,t]\times D}x\{M-N\}(ds\,dx)\qquad(t\in\boldsymbol{T}).$$

Therefore we see by (iii) that $\boldsymbol{\Theta}=^d[b, N]$ with respect to $\tilde{\boldsymbol{Q}}$. This implies that $\boldsymbol{P}_X=[\tilde{\boldsymbol{P}}]_{\boldsymbol{\Theta}}$ and $\boldsymbol{P}_Y=[\tilde{\boldsymbol{Q}}]_{\boldsymbol{\Theta}}$ on $\mathbf{D}(\boldsymbol{T})$, where $[\tilde{\boldsymbol{P}}]_{\boldsymbol{\Theta}}$ stands for the image of $\tilde{\boldsymbol{P}}$ induced by the map Θ: $\tilde{\Omega}\to\mathbf{D}(\boldsymbol{T})$. Thus $\tilde{\boldsymbol{P}}\sim\tilde{\boldsymbol{Q}}$ implies $\boldsymbol{P}_X\sim\boldsymbol{P}_Y$, which completes the proof of the "if" part.

5.3. *The "only if" part of Theorem* 1. Suppose that $\boldsymbol{P}_X\sim\boldsymbol{P}_Y$ is true. For each $\phi\in\mathbf{D}(\boldsymbol{T})$ and $U\in\mathscr{B}(\boldsymbol{S})$, we denote by $(J\phi)(U)$ the number of points $t\in\boldsymbol{T}$ satisfying the condition $(t,\phi(t)-\phi(t-))\in U$. Then we have a measurable map J: $\mathbf{D}(\boldsymbol{T})\to\mathscr{N}$ and a Poisson random measure $\{J(U,X)=(JX)(U);\ U\in\mathscr{B}(\boldsymbol{S})\}$ on $\boldsymbol{S}$ with intensity M. We note that

(5.10) $$\int_{\mathscr{N}}\exp(-\langle\mu,f\rangle)\,[\boldsymbol{P}_X]_J(d\mu)=\exp\left[-\iint_{\boldsymbol{S}}\{1-\exp(-f(s,x))\}\,M(ds\,dx)\right]$$

holds for any $f\in\mathscr{F}^+(\boldsymbol{S})$, where $[\boldsymbol{P}_X]_J$ stands for the image of $\boldsymbol{P}_X$ induced by the map J. Now we see that $\boldsymbol{P}_X\sim\boldsymbol{P}_Y$ implies $[\boldsymbol{P}_X]_J\sim[\boldsymbol{P}_Y]_J$ on $\mathscr{N}$. Then we obtain both (i) and (ii) by the result of Takahashi [14]. The condition (iii) is obtained by the same technique that was employed by Brockett and Tucker [2]. Therefore we complete the proof of the "only if" part. In the above discussion we see any violation of (i)$-$(iii) implies $\boldsymbol{P}_X\perp\boldsymbol{P}_Y$. Thus we obtain either $\boldsymbol{P}_X\sim\boldsymbol{P}_Y$ or $\boldsymbol{P}_X\perp\boldsymbol{P}_Y$ under the hypothesis $M\sim N$.

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